Handout 6 - The Armington model and Gravity Equations

Tiago Tavares*
International Trade I at ITAM

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1 Armington model and trade gravity

The Armington model assumes that consumers value not just domestically produced goods, but also foreign goods. We will see that this kind of structure is of particular importance to derive a very famous statistical representation of trade flows named gravity equation. This (empirical) equation states that a country tends to export more to countries that are larger in income, with that relationship being negatively affected by distance or other trade barriers. Because of the very good adherence of this statistical relationship, economists have used it to estimate the impact of policies aimed to reduce barriers to trade on imports, exports, and so on. However, such policies can also have an indirect impact on general prices that in turn also affect trade. The Armington model (which is a simple extension of a Ricardian model) underlines those indirect channels allowing us to derive a more rich analysis of the impact of such policies.

1.1 A two country Armington model

Suppose we have a world with only two countries $i = Home, Foreign$. Each country can only produce in a sector that is specific to that country, nevertheless consumers in both countries value diversity so they actually value consumption from all sectors in the world meaning that they are required to import goods from abroad in order to sustain

*Email: Please contact me to tgstavares@gmail.com if you find any errors in the text.

consumption of goods that are produced both at home and abroad. Labor, in fixed supply, is the only factor of production used linearly in both countries with a specific productivity of z_i. Moreover, we assume that an iceberg cost exists when goods are shipped to a foreign country. This means that $\tau_{hf} > 1$ is the iceberg cost of shipping goods the home economy to the foreign economy and τ_{fh} is the equivalent cost for goods shipped from the foreign to the domestic economy. We assume that $\tau_{hh} = \tau_{ff} = 1$, that is, there's no cost of transporting goods that are produced and consumed in the same economy. We characterize the competitive world market equilibrium by first describing supply and them demand for both countries and finally imposing some market clearing conditions that will allow us to understand these gravity equations.

1.1.1 Supply

Because markets are competitive, the cost of producing a single good is given by $w_i \cdot 1/z_i$ or, in words, the wage rate multiplied by the labor required to produce one unit of of the good. Because of an iceberg cost, the cost at the destination of a good produced in the home country to be consumed at the foreign country is given by $w_i \cdot \tau_{ij}/z_i$. Equating these marginal costs to the final price gives:

$$p_{ij} = w_i \cdot \tau_{ij}/z_i \tag{1}$$

(for example, $p_{hf} = w_h \cdot \tau_{hf}/z_h$ and $p_{hh} = w_h \cdot \tau_{hh}/z_h = w_h/z_h$ since $\tau_{hh} = 1$). Given the fixed supply of labor, output is produced either be for home consumption, x_{ii}, or foreign consumption, x_{ij}, that is:

$$L_i = x_{ij} \cdot \tau_{ij}/z_i + x_{ii} \cdot \tau_{ii}/z_i \tag{2}$$

where the first term in the RHS of the equation is labor required to produce exports and the second the labor required to produce domestic consumption.

1.1.2 Demand

As mentioned before, consumers value goods produced in both the home and foreign economy accordingly to a constant elasticity of substitution utility function[1]:

$$C_i = \left(\alpha_{ii}^{1/\sigma} x_{ii}^{\frac{\sigma-1}{\sigma}} + \alpha_{ji}^{1/\sigma} x_{ji}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{3}$$

[1] It will become apparent below why we use the notation C_i to denote total utility instead of the traditional U_i.

where x_{ji} is the consumption in country i of a good produced in country j; $\alpha_{ji} > 0$ is a parameter that reflects the preferences of country i towards goods produced in country j; and $\sigma > 0$ is a parameter that reflect the elasticity of substitution between good x_{ii} and x_{ji}[2]. This consumer wishes to minimize her expenditure by solving the following problem:

$$E\left(p_{ii}, p_{ji}, \bar{C}\right) = \min_{x_{ii}, x_{ji}} \left\{ p_{ii}x_{ii} + p_{ji}x_{ji} \quad subject\, to: \; \left(\alpha_{ii}^{1/\sigma} x_{ii}^{\frac{\sigma-1}{\sigma}} + \alpha_{ji}^{1/\sigma} x_{ji}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \geq \bar{C} \right\} \quad (4)$$

This essential means that, subject to a unitary utility level, consumers wish to minimize expenditure from goods produced domestically and abroad. As usual, first order conditions imply:

$$p_{ji} = \lambda_i \alpha_{ji}^{1/\sigma} x_{ji}^{-1/\sigma} C_i^{1/\sigma} \quad j = i, j \quad (5)$$

or, rearranging:

$$x_{ij} = \alpha_{ji} \left(\frac{p_{ji}}{\lambda_i}\right)^{-\sigma} C_i \quad (6)$$

where λ_i is a Lagrangian multiplier associated with the constraint and C_i is just the aggregate utility level (from consumption of both goods). Dividing the optimal p_{ii} by p_{ji} yields:

$$\frac{p_{ii}}{p_{ji}} = \left(\frac{\alpha_{ii}}{\alpha_{ji}}\right)^{1/\sigma} \left(\frac{x_{ii}}{x_{ji}}\right)^{-1/\sigma}$$

$$\Rightarrow \frac{x_{ii}}{x_{ij}} = \left(\frac{p_{ii}}{p_{ji}}\right)^{-\sigma} \cdot \frac{\alpha_{ii}}{\alpha_{ji}} \quad (7)$$

This equation allows us to see that the elasticity of substitution between good x_{ii} and x_{ij} is precisely σ. That is, if we take logs of equation (7) and differentiate we get:

$$\log\left(\frac{x_{ii}}{x_{ij}}\right) = -\sigma$$

$$\Rightarrow \frac{\partial \log\left(\frac{x_{ii}}{x_{ij}}\right)}{\partial \log\left(\frac{p_{ii}}{p_{ji}}\right)} = -\sigma$$

[2] That the elasticity of substitution, defined as $\frac{\partial \log(x_{ii}/x_{ji})}{\partial \log(p_{ii}/p_{ij})}$, is equal to σ can easily seen from the foc of the consumer problem as shown below on this note.

Now, substituting the equilibrium conditions in the (6) in the utility constrain of the cost minimization problem, allow us to solve for λ_i:

$$\begin{aligned}
\left(\alpha_{ii}^{1/\sigma} x_{ii}^{\frac{\sigma-1}{\sigma}} + \alpha_{ji}^{1/\sigma} x_{ji}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} &= C_i \\
\Rightarrow \left(\alpha_{ii}^{1/\sigma}\left(\alpha_{ii}\left(\frac{p_{ii}}{\lambda_i}\right)^{-\sigma} C_i\right)^{\frac{\sigma-1}{\sigma}} + \alpha_{ji}^{1/\sigma}\left(\alpha_{ji}\left(\frac{p_{ji}}{\lambda_i}\right)^{-\sigma} C_i\right)^{\frac{\sigma-1}{\sigma}}\right) &= C_i^{\frac{\sigma-1}{\sigma}} \\
\Rightarrow \qquad \left(\alpha_{ii}^{1/\sigma}\left(\alpha_{ii}\left(p_{ii}\right)^{-\sigma}\right)^{\frac{\sigma-1}{\sigma}} + \alpha_{ji}^{1/\sigma}\left(\alpha_{ji}\left(p_{ji}\right)^{-\sigma}\right)^{\frac{\sigma-1}{\sigma}}\right) &= \lambda_i^{1-\sigma} C_i^{\frac{\sigma-1}{\sigma}} \\
\Rightarrow \qquad \left(\alpha_{ii}\left(p_{ii}\right)^{1-\sigma} + \alpha_{ji}\left(p_{ji}\right)^{1-\sigma}\right) \cdot C_i^{\frac{1-\sigma}{\sigma}} &= \lambda_i^{1-\sigma} \\
\Rightarrow \qquad \left(\alpha_{ii}\left(p_{ii}\right)^{1-\sigma} + \alpha_{ji}\left(p_{ji}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \cdot C_i^{\frac{1}{\sigma}} &= \lambda_i
\end{aligned} \tag{8}$$

We are now in conditions to derive our expenditure function given by the substitution of our equilibrium conditions (6) and (8) into the cost minimization problem (4).

$$\begin{aligned}
E\left(p_{ii}, p_{ji}, \bar{C}\right) &= p_{ii} x_{ii} + p_{ji} x_{ji} \\
\Rightarrow E\left(p_{ii}, p_{ji}, \bar{C}\right) &= p_{ii}\alpha_{ii}\left(\frac{p_{ii}}{\lambda_i}\right)^{-\sigma} \bar{C} + p_{ji}\alpha_{ji}\left(\frac{p_{ji}}{\lambda_i}\right)^{-\sigma} \bar{C} \\
\Rightarrow E\left(p_{ii}, p_{ji}, \bar{C}\right) &= p_{ii}\alpha_{ii}\left(\frac{p_{ii}}{\lambda_i}\right)^{-\sigma} \bar{C} + p_{ji}\alpha_{ji}\left(\frac{p_{ji}}{\lambda_i}\right)^{-\sigma} \bar{C} \\
\Rightarrow E\left(p_{ii}, p_{ji}, \bar{C}\right) &= p_{ii}^{1-\sigma}\alpha_{ii}\lambda_i^{\sigma}\bar{C} + p_{ji}^{1-\sigma}\alpha_{ji}\lambda_i^{\sigma}\bar{C}
\end{aligned}$$

Given this expenditure function, we can measure the required expenditure necessary to deliver 1 unit of aggregate utility by setting $\bar{C} = 1$ and using the definition of λ_i:

$$\begin{aligned}
E\left(p_{ii}, p_{ji}, 1\right) &= p_{ii}^{1-\sigma}\alpha_{ii}\lambda_i^{\sigma} + p_{ji}^{1-\sigma}\alpha_{ji}\lambda_i^{\sigma} \\
\Rightarrow E\left(p_{ii}, p_{ji}, 1\right) &= \lambda_i^{\sigma}\left(p_{ii}^{1-\sigma}\alpha_{ii} + p_{ji}^{1-\sigma}\alpha_{ji}\right) \\
\Rightarrow E\left(p_{ii}, p_{ji}, 1\right) &= \lambda_i^{\sigma}\left(p_{ii}^{1-\sigma}\alpha_{ii} + p_{ji}^{1-\sigma}\alpha_{ji}\right) \\
\Rightarrow E\left(p_{ii}, p_{ji}, 1\right) &= \lambda_i^{\sigma}\lambda_i^{1-\sigma} \\
\Rightarrow E\left(p_{ii}, p_{ji}, 1\right) &= \lambda_i
\end{aligned}$$

The last equation give us an economic interpretation for λ_i as the expenditure required to achieve one unit of aggregate utility. It is standard in the literature to refer to that particular expenditure level as perfect price index P_i:

$$P_i \equiv E\left(p_{ii}, p_{ji}, 1\right) = \lambda_i = \left(\alpha_{ii}\left(p_{ii}\right)^{1-\sigma} + \alpha_{ji}\left(p_{ji}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \tag{9}$$

Going back to the function (6), we are able to re-write it as:

$$x_{ij} = \alpha_{ji}\left(\frac{p_{ji}}{P_i}\right)^{-\sigma} C_i \tag{10}$$

which has a typical demand function interpretation: demand for good x_{ij} is decreasing with its own price p_{ji} (a direct price effect), increases when the overall price P_i of all other goods increase (this is due to a substitution effect), and increases when the aggregate consumption C_i increases (this is due to an income effect).

To conclude this section, we present another interpretation for the optimal expenditure of a particular good as a function of the model parameters. To do that, we use the optimal condition (7) and then we sum over all goods:

$$\begin{aligned}
\frac{x_{ii}}{x_{ij}} &= \left(\frac{p_{ii}}{p_{ji}}\right)^{-\sigma} \cdot \frac{\alpha_{ii}}{\alpha_{ji}} \\
\Rightarrow \quad x_{ii}p_{ji}^{-\sigma}\alpha_{ji} &= \alpha_{ii}x_{ij}p_{ii}^{-\sigma} \\
\Rightarrow \quad x_{ii}p_{ii}p_{ji}^{1-\sigma}\alpha_{ji} &= \alpha_{ii}x_{ji}p_{ji}p_{ii}^{1-\sigma} \\
\Rightarrow x_{ii}p_{ii}p_{ji}^{1-\sigma}\alpha_{ji} + \alpha_{ii}p_{ii}^{1-\sigma}x_{ii}p_{ii} &= \alpha_{ii}p_{ii}^{1-\sigma}x_{ji}p_{ji} + \alpha_{ii}p_{ii}^{1-\sigma}x_{ii}p_{ii} \\
\Rightarrow \quad x_{ii}p_{ii}\left(p_{ji}^{1-\sigma}\alpha_{ji} + \alpha_{ii}p_{ii}^{1-\sigma}\right) &= \alpha_{ii}p_{ii}^{1-\sigma}\left(x_{ji}p_{ji} + x_{ii}p_{ii}\right) \\
\Rightarrow \quad x_{ii}p_{ii} &= \alpha_{ii}\frac{p_{ii}^{1-\sigma}}{\left(p_{ji}^{1-\sigma}\alpha_{ji} + \alpha_{ii}p_{ii}^{1-\sigma}\right)}\left(x_{ji}p_{ji} + x_{ii}p_{ii}\right) \\
\Rightarrow \quad x_{ii}p_{ii} &= \alpha_{ii}\left(\frac{p_{ii}}{P_i}\right)^{1-\sigma}\left(x_{ji}p_{ji} + x_{ii}p_{ii}\right) \\
\Rightarrow \quad x_{ii}p_{ii} &= \alpha_{ii}\left(\frac{p_{ii}}{P_i}\right)^{1-\sigma} X_i
\end{aligned}$$

and similarly to $x_{ji}p_{ji}$:

$$x_{ji}p_{ji} = \alpha_{ji}\left(\frac{p_{ji}}{P_i}\right)^{1-\sigma} X_i \tag{11}$$

where $X_i = x_{ji}p_{ji} + x_{ii}p_{ii}$ is the aggregate expenditure of country i. Note that the expenditure function in a particular good is a positive function of aggregate expenditure (which needs to be equal to income in equilibrium). Another feature of these expenditure function is that it is very similar to the demand function that we derived on (10). To see what are their relationship let's multiply (10) by p_{ji} to get:

$$\begin{aligned} x_{ji}p_{ji} =& p_{ji}\alpha_{ji}\left(\frac{p_{ji}}{P_i}\right)^{-\sigma} C_i \\ \Rightarrow x_{ji}p_{ji} =& \alpha_{ji}\left(\frac{p_{ji}}{P_i}\right)^{1-\sigma} \underbrace{P_iC_i}_{X_i} \end{aligned}$$

That is, one unit of aggregate consumption (which is in fact one unit of aggregate utility) evaluated at the aggregate perfect price index must equal total expenditure in this economy:

$$P_iC_i = X_i \equiv p_{ii}x_{ii} + p_{ji}x_{ji}$$

Another feature from (11) is that expenditure in a particular good can either be positive of negatively related with it's own price depending on σ. If σ is very large $\sigma > 1$, reflecting strong substitution effect between goods, then a price increase induces a strong substitution between that good and the other goods that leads to a fall in the expenditure on that good. However, if $\sigma < 1$, we have now stronger complementarities between goods, implying that an increase in a good price induces a very moderate fall in consumption of that good due to the fact that the consumer cannot find other goods that are close substitutes, inducing an increase in that good expenditure.

1.1.3 Gravity

We now want to relate how does the income between the home and foreign economy and it's distance influences the intensity of trade between the two countries. That relationship is precisely at the crux of the so called gravity equations.

We start with our expenditure equation of imports from the home economy as in equation (11):

$$x_{fh}p_{fh} = \alpha_{fh}\left(\frac{p_{fh}}{P_h}\right)^{1-\sigma} X_h$$

and because, from the firms optimization problem (1), $p_{fh} = w_f \cdot \tau_{fh}/z_f$, we can substitute to get:

$$
\begin{aligned}
x_{fh}p_{fh} =&\alpha_{fh}\left(\frac{w_f \cdot \tau_{fh}/z_f}{P_h}\right)^{1-\sigma} X_h \\
\Rightarrow x_{fh}p_{fh} =&\alpha_{fh} w_f^{1-\sigma}\left(\frac{\tau_{fh}/z_f}{P_h}\right)^{1-\sigma} X_h
\end{aligned} \tag{12}
$$

Note that, as usual, the wage rate in the foreign economy is determined from market clearing conditions (labor income equals expenditure in goods):

$$
\begin{aligned}
X_f &= w_f L_f \\
&= w_f\left(x_{ff}\frac{\tau_{ff}}{z_f} + x_{fh}\frac{\tau_{fh}}{z_f}\right) \\
&= x_{ff}w_f\frac{\tau_{ff}}{z_f} + x_{fh}w_f\frac{\tau_{fh}}{z_f} \\
&= x_{ff}p_{ff} + x_{fh}p_{fh} \\
&= \alpha_{ff}\left(\frac{p_{ff}}{P_f}\right)^{1-\sigma} X_f + \alpha_{fh}\left(\frac{p_{fh}}{P_h}\right)^{1-\sigma} X_h \\
&= \alpha_{ff}\left(\frac{w_f\tau_{ff}/z_f}{P_f}\right)^{1-\sigma} X_f + \alpha_{fh}\left(\frac{w_f\tau_{fh}/z_f}{P_h}\right)^{1-\sigma} X_h \\
&= w_f^{1-\sigma}\left[\alpha_{ff}\left(\frac{\tau_{ff}/z_f}{P_f}\right)^{1-\sigma} X_f + \alpha_{fh}\left(\frac{\tau_{fh}/z_f}{P_h}\right)^{1-\sigma} X_h\right]
\end{aligned}
$$

which allows to get an expression for the wages that equals:

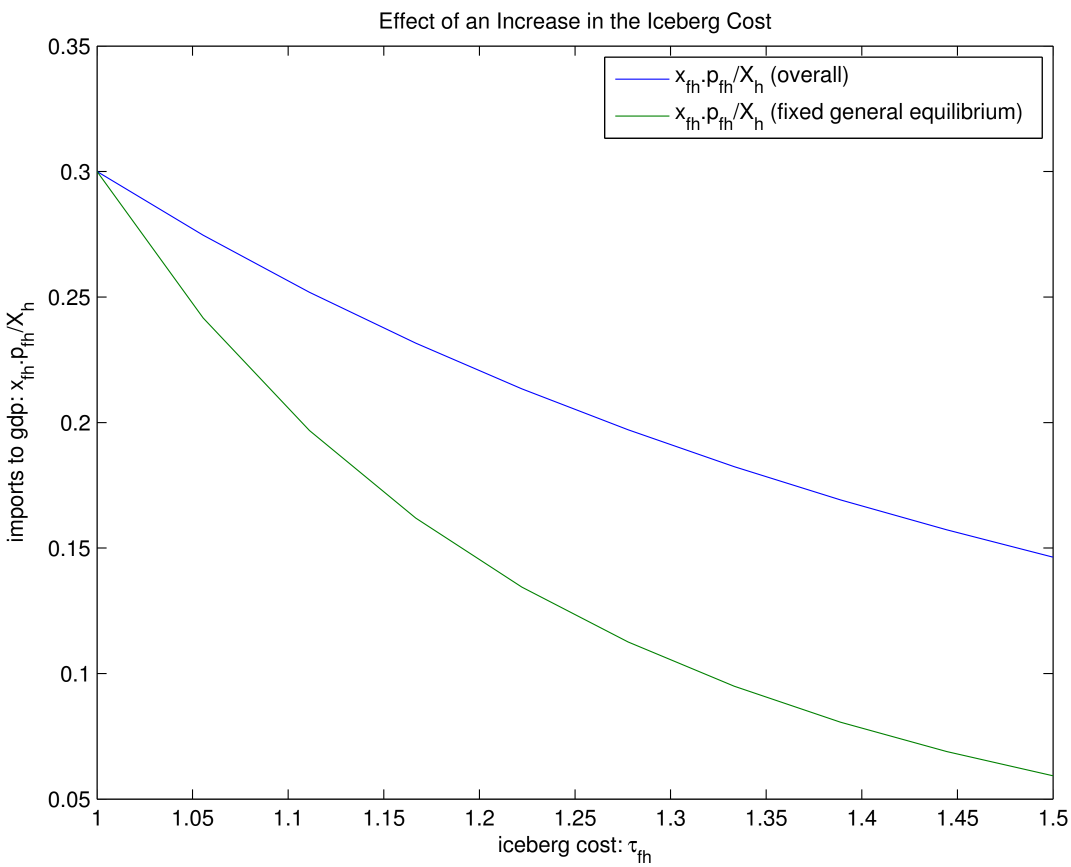
$$
w_f^{1-\sigma} = \frac{X_f}{\alpha_{ff}\left(\frac{\tau_{ff}/z_f}{P_f}\right)^{1-\sigma} X_f + \alpha_{fh}\left(\frac{\tau_{fh}/z_f}{P_h}\right)^{1-\sigma} X_h}
$$

Note that we can now substitute $w_f^{1-\sigma}$ in (12) to finally get our main gravity equation:

$$
\begin{aligned}
x_{fh}p_{fh} =&\alpha_{fh}\frac{X_f}{\alpha_{ff}\left(\frac{\tau_{ff}/z_f}{P_f}\right)^{1-\sigma} X_f + \alpha_{fh}\left(\frac{\tau_{fh}/z_f}{P_h}\right)^{1-\sigma} X_h}\left(\frac{\tau_{fh}/z_f}{P_h}\right)^{1-\sigma} X_h \\
\Rightarrow \underbrace{x_{fh}p_{fh}}_{\textit{imports home}} =&\underbrace{X_f X_h\left(\tau_{fh}/z_f\right)^{1-\sigma}}_{\textit{gravity}} \cdot \underbrace{\frac{P_h^{\sigma-1}\alpha_{fh}}{\alpha_{ff}\left(\frac{\tau_{ff}/z_f}{P_f}\right)^{1-\sigma} X_f + \alpha_{fh}\left(\frac{\tau_{fh}/z_f}{P_h}\right)^{1-\sigma} X_h}}_{\textit{general equilibrium}}
\end{aligned} \tag{13}
$$

It follows that the gravity equation can be decomposed into 2 main components: gravity and a general equilibrium term. The **gravity** term just shows that large economies, that is tend to spend more in imports due to the fact they they also have larger incomes (this is the $X_f X_h$ term). This effect, however, wanes away when distance increases as then, the effective price of an import increases and countries start substituting for domestic production (this is the τ_{fh}/z_f term). The impact of distance on trade is larger the larger is the degree to which goods can be substituted captured by the parameter σ, implying that a larger σ would imply a larger negative impact of trade from increasing distances. It is important to note that, here, distance τ_{fh} should be interpreted in a broad context that can include many barriers such as tariffs, physical distance, language, sharing a common border, belonging to a trade treaty region, etc. The **general equilibrium** effect shows that any exogenous change in distance or income should also have another, indirect impact through the change in overall prices P_h and P_f and a additional interaction term with income itself. For example, a decrease in the transportation cost would imply a direct positive impact in trade through gravity and an indirect effect through general equilibrium effects where, because of the fall in price of imports, the overall price index P_h also fall, implying that because the price of all other goods are now relatively cheaper, then the incentive to import is no longer that strong. As shown in a simulation of the model in figure 1.

Figure 1: The effect of the increase of τ_{fh} from 1 to 1.5 in an world economy with $L_h = L_f = 1$, $z_h = z_f = 1$, $\tau_{hf} = 1$, $\alpha_{hh} = \alpha_{ff} = 0.7$ and $\alpha_{fh} = \alpha_{hf} = 0.3$



It has been show that this gravity equation has a very strong adherence with the data. In many introductory textbooks such equation is simplified to:

$$T_{ij} = A \cdot Y_i \cdot Y_j / D_{ij}$$

where T_{ij} is the imports of country j to country i; Y_i and Y_j are the GDP of country i and j; D_{ij} is the distance between i and j; A is a constant (in our model is the last term of 13 which represents a general equilibrium effect). The empirical estimates of this equation are a great starting point to make a prediction the impact of a policy that reduces trade barriers (by reducing the effective D_{ij}) on the import intensity. However, it should also be noted that such estimates consider that A is constant when, from our discussion in the previous paragraph, in fact general equilibrium effects may tamper the impact of the cost reduction. An alternative analysis that has become popular among economists is to structural estimate versions of the model that we described before that takes into account explicitly the indirect effects on overall price indexes.

We should note that this 2 country model can be easily to one with an arbitrary number of countries without loosing any of the main conclusions. That is done in the following section.

1.2 A N country Armington model

Let be N countries indexed by i. We assume that each country has L_i consumers.

1.2.1 Supply

Markets are competitive and the price for each unit of the good produced is w_i. Transportation costs are modeled as an "iceberg cost" τ_{ij}, representing the cost of shipping goods from country i to country j[3]. The implication of such costs is that the final price of a good at destination is given by:

$$p_{ij} = w_i \tau_{ij}$$

Given the supply of labor, total output is given by:

$$L_i = \sum_{j=1}^{N} \tau_{ij} x_{ij}$$

where x_{ij} is the total quantity demanded by country j.

1.2.2 Demand

The representative consumer in country i has preferences characterized by the following constant elasticity of substitution utility function:

$$C_i = \left(\sum_{j=1}^{N} \alpha_{ji}^{1/\sigma} x_{ji}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}$$

[3]That is, in order to 1 unit of good to arrive to country j, country i has to ship τ_{ij} goods. Note that the model doesn't have productivity differences across countries, but the concept of τ_{ij} can easily absorb productivity differences.

where x_{ji} is the consumption level of country i of a good produced in county j; $\alpha_{ij} > 0$ is a preference parameter; and $\sigma \in [0, \infty]$ is the elasticity of substitution.

The problem for a consumer in country i consists in minimizing expenditure for a particular level of utility:

$$\min_{\{x_{ji}\}_{j=1,N}} \left\{ \sum_{j=1}^{N} p_{ji} x_{ji} \quad subject\, to: \; \left(\sum_{j=1}^{N} \alpha_{ji}^{1/\sigma} x_{ji}^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)} \geq \bar{C} \right\}$$

Thus, first order conditions imply:

$$p_{ji} = \lambda_i \alpha_{ji}^{1/\sigma} x_{ji}^{-1/\sigma} C_i^{1/\sigma} \quad j = 1, N \tag{14}$$

where λ is a Lagrange multiplier. Thus for goods originating from country j and k we have:

$$\begin{aligned} \frac{p_{ji}}{p_{ki}} &= \left(\frac{\alpha_{ji}}{\alpha_{ki}} \right)^{1/\sigma} \left(\frac{x_{ji}}{x_{ki}} \right)^{-1/\sigma} \\ \Rightarrow \qquad \frac{x_{ji}}{x_{ki}} &= \frac{\alpha_{ji}}{\alpha_{ki}} \left(\frac{p_{ji}}{p_{ki}} \right)^{-\sigma} \\ \Rightarrow \alpha_{ki} p_{ki}^{1-\sigma} x_{ji} p_{ji} &= \alpha_{ji} p_{ji}^{1-\sigma} x_{ki} p_{ki} \end{aligned}$$

Thus, summing over $k = 1, ...N$:

$$\begin{aligned} \alpha_{ki} p_{ki}^{1-\sigma} x_{ji} p_{ji} &= \alpha_{ji} p_{ji}^{1-\sigma} x_{ki} p_{ki} \\ \Rightarrow x_{ji} p_{ji} \sum_k \alpha_{ki} p_{ki}^{1-\sigma} &= \alpha_{ji} p_{ji}^{1-\sigma} \sum_k x_{ki} p_{ki} \\ \Rightarrow x_{ji} p_{ji} \sum_k \alpha_{ki} p_{ki}^{1-\sigma} &= \alpha_{ji} p_{ji}^{1-\sigma} \sum_k x_{ki} p_{ki} \\ \Rightarrow \qquad x_{ji} p_{ji} &= \alpha_{ji} \frac{p_{ji}^{1-\sigma}}{\sum_k \alpha_{ki} p_{ki}^{1-\sigma}} \sum_k x_{ki} p_{ki} \\ \Rightarrow \qquad x_{ji} p_{ji} &= \alpha_{ji} \left(\frac{p_{ji}}{P_i} \right)^{1-\sigma} X_i \end{aligned} \tag{15}$$

where the expenditure level in country i is:

$$X_i = \sum_k x_{ki} p_{ki}$$

and the perfect price index in country i is:

$$P_i = \left(\sum_k \alpha_{ki} p_{ki}^{1-\sigma} \right)^{1/(1-\sigma)}$$

One should note that using the the expenditure in this economy i equals C_i. To see this, substitute 14 in the objective function and rearrange:

$$\begin{aligned} p_{ji} =& \lambda \alpha_{ji}^{1/\sigma} x_{ji}^{-1/\sigma} C_i^{1/\sigma} \\ \Rightarrow p_{ji} x_{ji} =& \lambda^\sigma \alpha_{ji}^\sigma p_{ji}^{1-\sigma} C_i \end{aligned}$$

thus:

$$\begin{aligned} \sum_k p_{ki} x_{ki} =& P_i \\ \Rightarrow \lambda^\sigma C_i \sum_k \alpha_{ki}^\sigma p_{ki}^{1-\sigma} =& C_i P_i \\ \Rightarrow \qquad \lambda^\sigma =& \frac{P_i}{\sum_k \alpha_{ki}^\sigma p_{ki}^{1-\sigma}} \\ \Rightarrow \qquad \lambda^\sigma =& \left(\sum_k \alpha_{ki} p_{ki}^{1-\sigma} \right)^{1/(1-\sigma)-1} \\ \Rightarrow \qquad \lambda^\sigma =& \left(\sum_k \alpha_{ki} p_{ki}^{1-\sigma} \right)^{\sigma/(1-\sigma)} \\ \Rightarrow \qquad \lambda =& P_i \end{aligned}$$

Note that this also shows that the Lagrange multiplier associated with utility is the same as the perfect price index, that is, P_i is the derivative of expenditure with respect to the constraint, and yet another interpretation for P_i is necessary expenditure to consume the equivalent one unit of utility.

1.2.3 Gravity

Note that the share of consumption of goods produced in country i in country j is given by:

$$\begin{aligned}\theta_{ij} &= \frac{x_{ij}p_{ij}}{\sum_k x_{kj}p_{kj}} \\ &= \frac{\alpha_{ij}\left(\frac{p_{ij}}{P_j}\right)^{1-\sigma}X_j}{\sum_k \alpha_{kj}\left(\frac{p_{kj}}{P_j}\right)^{1-\sigma}X_j} \\ &= \frac{\alpha_{ij}p_{ij}^{1-\sigma}}{\sum_k \alpha_{kj}p_{kj}^{1-\sigma}} \\ &= \frac{\alpha_{ij}w_i^{1-\sigma}\tau_{ij}^{1-\sigma}}{\sum_k \alpha_{kj}w_k^{1-\sigma}\tau_{ki}^{1-\sigma}}\end{aligned}$$

Recall that from the budget constraint of the consumer and trade balance:

$$X_i = w_iL_i = w_i\sum_{j=1}^{N}\tau_{ij}x_{ij} = \sum_{j=1}^{N}p_{ij}x_{ij} \tag{16}$$

Now use the following definition:

$$p_{ij} = p_{ii}\tau_{ij}$$

and substitute and sum in (16):

$$\begin{aligned}X_i =& \sum_{j=1}^{N}p_{ij}x_{ij} \\ \Rightarrow \quad X_i =& \sum_{j=1}^{N}\alpha_{ij}\left(\frac{p_{ij}}{P_j}\right)^{1-\sigma}X_j \\ \Rightarrow \quad X_i =& \sum_{j=1}^{N}\alpha_{ij}\left(\frac{w_i\tau_{ij}}{P_j}\right)^{1-\sigma}X_j \\ \Rightarrow \quad X_i =& w_i^{1-\sigma}\sum_{j=1}^{N}\alpha_{ij}\left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma}X_j \\ \Rightarrow w_i^{1-\sigma} =& \frac{X_i}{\sum_{j=1}^{N}\alpha_{ij}\left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma}X_j}\end{aligned}$$

And finally, from (15):

$$
\begin{aligned}
x_{ij}p_{ij} =& \alpha_{ij}\left(\frac{p_{ij}}{P_j}\right)^{1-\sigma} X_j \\
\Rightarrow x_{ij}p_{ij} =& \alpha_{ij} w_i^{1-\sigma}\left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} X_j \\
\Rightarrow x_{ij}p_{ij} =& \alpha_{ij}\frac{X_i}{\sum_{k=1}^{N}\alpha_{ij}\left(\frac{\tau_{ik}}{P_k}\right)^{1-\sigma} X_k}\left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} X_j \\
\Rightarrow x_{ij}p_{ij} =& \alpha_{ij}\frac{X_i}{\sum_{k=1}^{N}\alpha_{ik}\left(\frac{\tau_{ik}}{P_k}\right)^{1-\sigma} X_k}\left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} X_j \\
\Rightarrow x_{ij}p_{ij} =& X_i X_j \left(\tau_{ij}\right)^{1-\sigma}\frac{P_j^{\sigma-1}\alpha_{ij}}{\sum_{k=1}^{N}\alpha_{ik}\left(\frac{\tau_{ik}}{P_k}\right)^{1-\sigma} X_k} \qquad (17)
\end{aligned}
$$

The last equation is a gravity equation which has the same form as the one derived before: it shows that bilateral trading is related with the product of GDPs of both countries (gravity), and the distance/trade costs.

A Appendix

A.1 Solving for the 2 country Armington model

Note that the 2 country Armington model includes 4 prices $(p_{hh}, p_{ff}, w_h, w_f)$ and four consumption quantities $(x_{hh}, x_{fh}, x_{ff}, x_{hf})$ which give us 8 unknowns. Because we are just interested in relative prices, we can normalize one price to reduce the number of unknowns of the model to 7. We then need to 7 equilibrium equations to solve for the model. These equations are the following:

$$
\begin{aligned}
x_{hh}p_{hh} =& \alpha_{hh}\left(\frac{p_{hh}}{P_h}\right)^{1-\sigma} L_h w_h \\
x_{fh}p_{fh} =& \alpha_{fh}\left(\frac{p_{fh}}{P_h}\right)^{1-\sigma} L_h w_h \\
x_{ff}p_{ff} =& \alpha_{ff}\left(\frac{p_{ff}}{P_f}\right)^{1-\sigma} L_f w_f \\
x_{hf}p_{hf} =& \alpha_{hf}\left(\frac{p_{hf}}{P_f}\right)^{1-\sigma} L_f w_f \\
L_h =& \frac{\tau_{hh}}{z_h}x_{hh} + \frac{\tau_{hf}}{z_h}x_{hf} \\
L_f =& \frac{\tau_{ff}}{z_f}x_{ff} + \frac{\tau_{fh}}{z_f}x_{fh} \\
p_{hf}x_{hf} =& p_{fh}x_{fh}
\end{aligned}
$$

where the first 4 equations are demand equations, the next 2 are labor market clearing conditions, and the last one a world market clearing conditions for goods (that exports equal imports in value). Solving simultaneously for this system of equations allow us to get the picture from figure 1.