Mock Final (based on last year's exam) - Proposed solution topics

International Trade I at ITAM

December 2015

1. Consider trade between $N \geq 2$ countries. A representative consumer in country j maximizes his utility given by:

$$U_j = \left(\sum_{i=1}^{N} n_i c_{ij}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad , \quad \sigma > 1$$

where c_{ij} is the consumption in country j of a good produced in country i. Assuming monopolistic competition, each firm in each country produces a single product which is sold domestically and exported. All products exported by country i to j sell for the same price p_{ij}. n_i is the number of varieties produced (and exported) by country i. Labor is the only factor of production and each country is endowed with $\bar{L}_i$ with nominal wage rate of w_j.

   (a) Setup and solve the consumer's utility maximization problem to show that demand is given by:

$$c_{ij} = \left(\frac{p_{ij}}{P_j} \right)^{-\sigma} \frac{w_j \bar{L}_j}{P_j}$$

   where $P_j = \left(\sum_i^N n_i p_{ij}^{1-\sigma} \right)^{1/(1-\sigma)}$

   As usual:

$$\max_{c_{ij}} \quad \left(\sum_{i=1}^{N} n_i c_{ij}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

$$st$$

$$\sum_{i=1}^{N} p_{ij} n_i c_{ij} = w_j \bar{L}_j$$

First order conditions imply:

$$n_i c_{ij}^{-1/\sigma} U_j^{1/\sigma} = \lambda_j p_{ij} n_i$$

where λ_j is a multiplier on the budget constraint. Rearranging the above expression gives:

$$c_{ij} = \left(\frac{p_{ij}}{\lambda_j^{-1}}\right)^{-\sigma} U_j$$

Now, we should note that λ_j is the marginal utility of income, implying that λ_{ij}^{-1} is the marginal expenditure of utility, that is the price index. Also recall that U_j can be interpreted as a unit of aggregate consumption C_j. Note that from the identity expenditure equals income $P_j C_j = w_j \bar{L}_j$, we have:

$$\begin{aligned} U_j &\equiv C_j = \frac{w_j \bar{L}_j}{P_j} \\ P_j &\equiv \lambda_j^{-1} \end{aligned}$$

It follows that

$$c_{ij} = \left(\frac{p_{ij}}{\lambda_j^{-1}}\right)^{-\sigma} U_j = \left(\frac{p_{ij}}{P_j}\right)^{-\sigma} \frac{w_j \bar{L}_j}{P_j}$$

As for the formula for the price index, we just use the demand function on the budget constraint:

$$\begin{aligned} \sum_{i=1}^{N} p_{ij} n_i c_{ij} &= w_j \bar{L}_j \\ \Rightarrow \sum_{i=1}^{N} p_{ij} n_i \left(\frac{p_{ij}}{P_j}\right)^{-\sigma} C_j &= P_j C_j \\ \Rightarrow \qquad P_j &= \left(\sum_{i=1}^{N} n_i p_{ij}^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \end{aligned}$$

(b) Suppose that every variety requires a labor input of $L_i = \alpha + \beta_i X_i$ to produce X_i. This implies that every variety produced in country i is sold at the same price. Setup and solve the firm's profit maximization problem to show that:

$$p_i = \frac{\sigma}{\sigma - 1} \beta_i w_i$$

Show that price is greater than the marginal cost. What is the mark-up?

Firms maximize profits:

$$\pi = p_{ij}x_{ij} - w_j L_j = p_{ij}x_{ij} - w_j\left(\alpha + \beta_i x_{ij}\right)$$

knowing the demand function, that is

$$\pi = p_{ij}\left(\frac{p_{ij}}{P_j}\right)^{-\sigma} C_j - w_j\left(\alpha + \beta_i\left(\frac{p_{ij}}{P_j}\right)^{-\sigma} C_j\right)$$

Equating to zero the first derivative of the above expression with respect to price:

$$(1-\sigma)\, p_{ij}^{-\sigma}\left(\frac{1}{P_j}\right)^{-\sigma} C_j + \sigma p_{ij}^{1-\sigma} w_j \beta_i \left(\frac{1}{P_j}\right)^{-\sigma} C_j = 0$$

and rearranging yields:

$$p_{ij} = p_j = \frac{\sigma}{\sigma - 1} w_j \beta_j$$

Where the markup is $\frac{\sigma}{\sigma-1}$ and is obvious that

(c) We assume that all firms earn zero profit. Use this to show that the number of products produced by country i is given by:

$$n_i = \frac{\bar{L}_i}{\sigma\alpha}$$

Lets first equate revenues and costs at the firms optimal price:

$$\begin{aligned} p_i x_i =& w_j\left(\alpha + \beta_i x_i\right) \\ \Rightarrow \frac{\sigma}{\sigma-1} w_j \beta_j x_i =& w_j\left(\alpha + \beta_i x_i\right) \\ \Rightarrow \qquad x_i =& \frac{\alpha(\sigma - 1)}{\beta_i} \end{aligned}$$

In terms of labor this requires:

$$L_i = \alpha + \beta_i x_i = \alpha\sigma$$

Finally from labor market equilibrium:

$$n_i L_i = \bar{L}_i \Rightarrow n_i = \frac{\bar{L}_i}{\alpha\sigma}$$

(d) Now we add trade costs into the model as an iceberg cost $\tau_{ij} > 1$, meaning that in order to import goods from country i, country j has to pay $p_{ij} = \tau_{ij} p_i$, where p_i is the price charged by firms in country i (derived in b.).

i. Show that total expenditure by country j on goods imported from country i equals:

$$T_{ij} \equiv n_i p_{ij} c_{ij} = n_i Y_j \left(\frac{\tau_{ij} p_i}{P_j} \right)^{1-\sigma}$$

where $Y_j = w_j \bar{L}_j$

Starting from the demand equation that we derived before:

$$\begin{aligned} c_{ij} &= \left(\frac{p_{ij}}{P_j} \right)^{-\sigma} \frac{w_j \bar{L}_j}{P_j} \\ \Rightarrow n_i p_{ij} c_{ij} &= n_i p_{ij} \frac{P_j}{P_j} \left(\frac{p_{ij}}{P_j} \right)^{-\sigma} \frac{w_j \bar{L}_j}{P_j} \\ \Rightarrow \quad T_{ij} &= n_i Y_j \left(\frac{p_{ij}}{P_j} \right)^{1-\sigma} \\ \Rightarrow \quad T_{ij} &= n_i Y_j \left(\frac{\tau_{ij} p_i}{P_j} \right)^{1-\sigma} \end{aligned}$$

ii. Use the above expression to derive the following gravity equation:

$$T_{ij} = \frac{\beta_i}{\alpha(\sigma - 1)} \left(\frac{Y_i Y_j}{p_i^\sigma} \right) \left(\frac{\tau_{ij}}{P_j} \right)^{1-\sigma}$$

Explain intuitively what is the effect of an increase of τ_{ij} on T_{ij}. How does that depend on the elasticity of substitution σ?

Starting from the above equation

$$
\begin{aligned}
T_{ij} =& n_i Y_j \left(\frac{\tau_{ij} p_i}{P_j}\right)^{1-\sigma} \\
\Rightarrow T_{ij} =& n_i Y_j \left(\frac{\tau_{ij} p_i}{P_j}\right)^{1-\sigma} \\
\Rightarrow T_{ij} =& n_i Y_j \frac{p_i}{p_i^\sigma} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \\
\Rightarrow T_{ij} =& n_i Y_j \frac{\sigma}{p_i^\sigma} \frac{w_i \beta_i}{\sigma - 1} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \\
\Rightarrow T_{ij} =& n_i Y_j \frac{\sigma}{p_i^\sigma} \frac{w_i \beta_i}{\sigma - 1} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \\
\Rightarrow T_{ij} =& \frac{\bar{L}_i}{\alpha\sigma} Y_j \frac{\sigma}{p_i^\sigma} \frac{w_i \beta_i}{\sigma - 1} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \\
\Rightarrow T_{ij} =& \frac{\beta_i}{\alpha(\sigma - 1)} Y_j Y_i \frac{Y_j Y_i}{p_i^\sigma} \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma}
\end{aligned}
$$

2. Consider the Melitz model with monopolistically competitive firms and two identical countries. Each firm produces only one variety, which it can sell domestically or export it. The price p, revenue r, and profit π are given by:

$$
p(\varphi) = \begin{cases} p_d(\varphi) = & \frac{\sigma}{\sigma-1} \cdot \frac{1}{\varphi}, \quad domestic\, market \\ p_x(\varphi) = & \frac{\sigma}{\sigma-1} \cdot \frac{\tau}{\varphi}, \quad foreign\, market \end{cases}
$$

$$
r_i(\varphi) = \begin{cases} r_d(\varphi) = & \left(\frac{\sigma}{\sigma-1} \cdot \varphi P\right)^{\sigma-1} R, \quad domestic\, market \\ r_x(\varphi) = & \left(\frac{\sigma}{\sigma-1} \cdot \frac{\varphi}{\tau} P\right)^{\sigma-1} R, \quad foreign\, market \end{cases}
$$

$$
\pi_i(\varphi) = \begin{cases} \pi_d(\varphi) = & \left(\frac{\sigma}{\sigma-1} \cdot \varphi P\right)^{\sigma-1} \frac{R}{\sigma} - f, \quad domestic\, market \\ r_x(\varphi) = & \left(\frac{\sigma}{\sigma-1} \cdot \frac{\varphi}{\tau} P\right)^{\sigma-1} \frac{R}{\sigma} - f_x, \quad foreign\, market \end{cases}
$$

Note that if a firm exports, total revenues becomes $r(\varphi) = r_d(\varphi) + r_x(\varphi)$ and profits $\pi(\varphi) = \pi_d(\varphi) + \pi_x(\varphi)$. P is the price index; R is the total revenue on all varieties; f and f_x are the fixed cost of production for domestic and export market; $\tau > 1$ is the iceberg cost. Wages are normalized to 1 and we assume there is free entry and firms die with probability δ. Let f_e be the fixed cost of entering the market, and $G(\varphi)$ the distribution from which φ is surveyed for potential entrants ($g(\varphi)$ is the pdf of that distribution). Assume further that φ^* and φ_x^* are the cut-off productivities of producing domestically and to export markets. The average productivity of firms operating in the market is $\tilde{\varphi}$.

(a) Show that the open equilibrium is the solution of a zero profit condition and a free entry condition given by:

$$\pi(\tilde{\varphi}) \equiv \bar{\pi} = \frac{\delta f_e}{1 - G(\varphi^*)} \tag{FE}$$

$$\pi(\tilde{\varphi}) \equiv \bar{\pi} = f\left(\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*}\right)^{\sigma-1} - 1\right) + prob_x f_x\left(\left(\frac{\tilde{\varphi}_x(\varphi^*)}{\varphi_x^*(\tilde{\varphi})}\right)^{\sigma-1} - 1\right) \tag{ZPC}$$

where

$$\begin{aligned} \varphi_x^*/\varphi^* &= \tau\left(f_x/f\right)^{1/(\sigma-1)} \\ prob_x &= \frac{1 - G(\varphi_x^*)}{1 - G(\varphi^*)} \end{aligned}$$

Depict a graphical representation of the equilibrium.

Starting from the (FE), we know that the value for a firm with productivity φ is:

$$V(\varphi) = \max\left\{0, \frac{\pi(\varphi)}{\delta}\right\}$$

So ex-ante, the value for a potential entrant in expectation:

$$EV = G(\varphi^*) + [1 - G(\varphi^*)]\,\pi(\tilde{\varphi})/\delta - f_e$$

Equating the above to zero and rearranging:

$$\pi(\tilde{\varphi}) \equiv \bar{\pi} = \frac{\delta f_e}{1 - G(\varphi^*)}$$

As for the ZPC, note that the average productivity firm with generate the following profits:

$$\begin{aligned}
\pi(\tilde{\varphi}) \equiv \bar{\pi} &= \pi_d(\tilde{\varphi}) + \frac{1-G(\varphi_x^*)}{1-G(\varphi^*)}\pi_x(\tilde{\varphi}_x) \\
&= (r_d(\tilde{\varphi})/\sigma - f) + \frac{1-G(\varphi_x^*)}{1-G(\varphi^*)}(r_d(\tilde{\varphi}_x)/\sigma - f_x) \\
&= f\left(\frac{r_d(\tilde{\varphi})}{f\sigma} - 1\right) + \frac{1-G(\varphi_x^*)}{1-G(\varphi^*)} f_x\left(\frac{r_x(\tilde{\varphi}_x)}{f_x\sigma} - 1\right)
\end{aligned}$$

But note that at the cut-off productivity $r_i(\varphi_i^*) = \sigma f_i$, so:

$$\begin{aligned}
\pi(\tilde{\varphi}) \equiv \bar{\pi} &= f\left(\frac{r_d(\tilde{\varphi})}{f\sigma} - 1\right) + \frac{1-G(\varphi_x^*)}{1-G(\varphi^*)} f_x\left(\frac{r_x(\tilde{\varphi}_x)}{f_x\sigma} - 1\right) \\
&= f\left(\frac{r_d(\tilde{\varphi})}{r_d(\varphi^*)} - 1\right) + \frac{1-G(\varphi_x^*)}{1-G(\varphi^*)} f_x\left(\frac{r_x(\tilde{\varphi}_x)}{r_d(\varphi_x^*)} - 1\right) \\
&= f\left(\left(\frac{\tilde{\varphi}}{\varphi^*}\right)^{\sigma-1} - 1\right) + \frac{1-G(\varphi_x^*)}{1-G(\varphi^*)} f_x\left(\left(\frac{\tilde{\varphi}_x}{\varphi_x^*}\right)^{\sigma-1} - 1\right) \\
&= f\left(\left(\frac{\tilde{\varphi}(\varphi^*)}{\varphi^*}\right)^{\sigma-1} - 1\right) + \frac{1-G(\varphi_x^*(\varphi^*))}{1-G(\varphi^*)} f_x\left(\left(\frac{\tilde{\varphi}_x(\varphi^*)}{\varphi_x^*(\varphi^*)}\right)^{\sigma-1} - 1\right)
\end{aligned}$$

Close inspection of the 2 equilibrium equations reveals that the (FE) is increasing while the (ZPC) is (usually) decreasing. That should be enough to determine the equilibrium.

(b) Show that higher productivity implies larger per worker revenues. You can use the fact that:

$$\begin{aligned}
\frac{r(\varphi)}{l(\varphi)} &= \frac{\sigma-1}{\sigma}\left(1 - \frac{f}{l(\varphi)}\right) \\
l(\varphi) &= f + q(\varphi)/\varphi \\
q(\varphi) &= r(\varphi)/p(\varphi)
\end{aligned}$$

In order to do this, we just need to understand the sign of the derivative $\partial\left[r(\varphi)/l(\varphi)\right]/\partial\varphi$. Note that

$$\frac{\partial r/l}{\partial \varphi} = \frac{\sigma-1}{\sigma}\frac{f}{[l(\varphi)]^2}\frac{\partial l}{\partial \varphi}$$

and

$$\begin{aligned}
\frac{\partial l}{\partial \varphi} &= \frac{\partial q}{\partial \varphi}/\varphi - q\left(\varphi\right)/\varphi^2 \\
&= P^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{\sigma} R\varphi^{\sigma}\sigma/\varphi^2 - P^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{\sigma} R\varphi^{\sigma}/\varphi^2 \\
&= P^{\sigma-1}\left(\frac{\sigma}{\sigma-1}\right)^{\sigma} R\frac{\varphi^{\sigma}}{\varphi^2}\left(\sigma-1\right) > 0
\end{aligned}$$

And the positive signal of the last derivative should prove our case.